Key Graph Theory Theorems

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3.3 Binary Trees

3.3.1 Problem (p.82) *Determine the number, t_n, of binary trees with n edges.*

The number of binary tress with n edges is

$$t_n = \frac{1}{n+1}\binom{2n}{n}$$

This is the *Catalan number*; these numbers arise frequently in enumeration.

4.1 Definitions

4.1.7 Handshake Theorem (p.99) *For a graph G having q edges, we have*

$$\sum_{v \in V(G)} \deg(v) = 2q$$

4.1.8 Corollary (p.99) *The number of vertices of odd degree in a graph is even.*

4.1.9 Definition (p.99) *The n-cube is the graph whose vertices are the 0,1-strings of length n, and two strings are adjacent if they differ in exactly one position, $n \geq 0$.*

4.1.11 Problem (p.100) *The n-cube has $p = 2^n$ vertices and $q = n2^{n-1}$ edges, for $n \geq 0$.*

4.1.12 Problem (p.101) *The n-cube is bipartite, for $n \geq 0$.*

4.3 Paths and Cycles

4.3.2 Theorem (p.109) *If there is a walk from vertex x to vertex y in G, then there is a path from x to y in G.*

4.3.3 Corollary (p.110) *Let x,y,z be vertices of G. If there if a path from x to y in G and a path from y to z in G, then there is a path from x to z in G.*

4.4 Connectedness

4.4.2 Theorem (p.112) *A graph G is connected if, for some fixed vertex v in G, there is a path from v to x in G for all other vertices x in G.*

4.4.3 Problem (p.112) *The n-cube is connected for each* $n \geq 0$.

4.4.4 Theorem (p.113) *A graph G is not connected if and only if there exists a proper nonempty subset X of V(G) such that the cut induced by X is empty.*

4.4.6 Lemma (p.114) *If e =* $\{x,y\}$ *is a bridge of a connected graph G, then G - e has precisely two components; furthermore, x and y are in different components.*

4.4.7 Theorem (p.114) *Edge e is a bridge of a graph G if and only if e is not in any cycle of G.*

4.4.8 Corollary (p.114) *If there are two distinct paths from vertex u to vertex v in G, then G contains a cycle.*

5.1 Trees

5.1.2 Lemma (p.118) *There is a unique path between every pair of vertices u and v in a tree T.*

5.1.3 Lemma (p.118) *Every edge of a tree T is a bridge.*

5.1.4 Theorem (p.118) *For any tree with p vertices and q edges,* $q = p - 1$.

5.1.5 Theorem (p.118) *A tree with at least two vertices has at least two vertices of degree one.*

Moreover,

$$n_1 = 2 + n_3 + 2n_4 + 3n_5 + 4n_6 + \cdots$$

where n_i = the number of vertices of degree i in a tree, $i \geq 0$.

5.2 Spanning Trees

5.2.1 Theorem (p.121) *A graph G is connected if and only if it has a spanning tree.*

5.2.2 Corollary (p.122) *If G is connected, with p vertices and q=p-1 edges, then G is a tree.*

5.2.4 Theorem (p.122) *A graph is bipartite if and only if it has no odd cycles.*

5.3 Breadth-First Search

5.3.1 Algorithm (p.123) *To find spanning tree of a graph G: Select any vertex r of G as the initial subgraph D, with $pr(r) = \emptyset$. At each stage, find an edge in G that joins a vertex u of D to a vertex v not in D. Add vertex v and edge u,v to D, with $pr(v)=u$. Stop when there is no such edge.*

Claim: If D contains p vertices when the algorithm terminates, then D is a spanning tree of G. If D contains less than p vertices when the algorithm terminates, then G is not connected (so, from Theorem 5.2.1, G has no spanning tree).

5.3.2 Algorithm (p.126) *Breadth-first search. Follow Algorithm 5.3.1 with the following refinement. At each stage consider the unexhausted vertices (called the* **active** *vertex), and choose an edge incident with this vertex and a vertex v not in the tree.*

5.3.3 Lemma (p.128) *The vertices enter a breadth-first search tree in non-decreasing order of level.*

5.3.4 Theorem (p.128) *(The primary property of breadth-first search.) In a connected graph with a breadth-first search tree, each non-tree edge in the graph joins vertices that are at most one level apart in the search tree (of course each tree edge joins vertices that are exactly one level apart).*

5.4 Applications of Breadth-First Search

5.4.1 Theorem (p.131) *A connected graph G with breadth-first search tree T has an odd cycle if and only if it has a non-tree edge joining vertices at the same level in T.*

5.4.2 Theorem (p.131) *The length of a shortest path from u to v in a connected graph G is equal to the level of v in any breadth-first search tree of G with u as the root.*

6.1 Planarity

6.1.2 Dual Handshake Theorem (p.151) *For a connected planar embedding with faces $f_1, \cdots, f_s$, we have*

$$\sum_{i=1}^{s} \deg(f_i) = 2q$$

6.2 Euler's Formula

6.2.1 Theorem (p.151) *For a planar embedding with p vertices, q edges, s faces, and c components, we have*

$$p - q + s = 1 + c$$

6.2.2 Theorem (p.152) *A graph is planar if and only if it can be drawn on the surface of a sphere.*

6.3 Platonic Solids

6.3.4 Theorem (p.156) *There are exactly five platonic graphs.*

6.3.2 Lemma (p.158) *Let G be a planar embedding with p vertices, q edges and s faces, in which each vertex has degree $d \geq 3$ and each face has degree $d^* \geq 3$. Then,*

$$(d, d^*) \in (3,3), (3,4), (4,3), (3,5), (5,3)$$

and

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}, \quad p = \frac{2q}{d}, \quad s = \frac{2q}{d^*}$$

6.4 Nonplanar Graphs

6.4.1 Lemma (p.159) *Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d^*, then $(d^* - 2)q \leq d^*(p-2)$.*

6.4.2 Lemma (p.160) *If G is a planar embedding that has at least one cycle, then the boundary of every face contains a cycle.*

6.4.3 Theorem (p.161) *In a planar graph on $p \geq 3$ vertices, we have*

$$q \leq 3p - 6$$

6.4.4 Corollary (p.161) *K_5 is a nonplanar graph.*

6.4.5 Corollary (p.162) *A planar graph has a vertex of degree at most 5.*

6.4.6 Theorem (p.163) *In a planar graph with a cycle, and girth at least k, for $k \geq 3$, we have*

$$q \leq \frac{k(p-2)}{k-2}$$

6.4.7 Corollary (p.163) *$K_{3,3}$ is a nonplanar graph.*

6.4.8 Kuratowski's Theorem (p.164) *A graph is nonplanar if and only if it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$.*

6.5 Colouring and Planar Graphs

6.5.2 Theorem (p.169) *A graph is 2-colourable if and only if it is bipartite.*

6.5.3 Theorem (p.170) K_p *is p-colourable, and not k-colourable for any* $k < p$.

Remark (p.171) *G/e is planar whenever G is. The converse is not true; G/e may be planar when G is non-planar.*

6.5.6 Theorem (p.172) *Every planar graph is 4-colourable.*

7.1 Matchings

7.1.1 Lemma (p.179) *If M admits an augmenting path, then M is not a maximum matching.*

7.1.2 Lemma (p.179) *If M is a matching of G and C is a cover of G, then* $|M| \leq |C|$.

7.1.3 Lemma (p.180) *If M is a matching and C is a cover and* $|M| = |C|$, *then M is a maximum matching and C is a minimum cover.*

7.2 König's Theorem

7.2.1 Theorem (p.182) *In a bipartite graph the maximum size of a matching is the minimum size of a cover.*

7.2.2 Lemma (p.183) *Let M be a matching of a bipartite graph G with bipartition A,B, and let X and Y be defined as a)* $X = A \cap Z$ *and b)* $Y = B \cap Z$. *Then:*

(a) There is no edge of G from X to $B \backslash Y$*;*

(b) $C = Y \cup (A \backslash X)$ *is cover of G;*

(c) There is no edge of M from Y to $A \backslash X$*;*

(d) $|M| = |C| - |U|$ *where U is the set of unsaturated vertices in Y;*

(e) There is an augmenting path to each vertex in U.

7.2.3 Problem (p.184) *If G is a bipartite graph with bipartition A, B, where* $|A| = |B| = n$, *then G has a matching of size at least* q/n.

Problem *If G is a bipartite graph with maximum degree k, then G has a matching of size at least* q/k.

7.3 Applications of König's Theorem

7.3.1 Hall's Theorem (p.189) *A bipartite graph G with bipartition A,B has a matching saturating every vertex in A, if and only if every subset D of A satisfies* $|N(D)| \geq |D|$.

7.3.2 Corollary (Hall's SDR Theorem) *The collection $Q_1, Q_2, \ldots Q_n$ of subsets of the finite set Q has an SDR if and only if, for every subset J of 1,2,...,n, we have*

$$|\bigcup_{i \in J} Q_i| \geq |J|$$

Perfect Matchings in Bipartite Graphs

7.3.3 Corollary (p.192) *A bipartite graph G with bipartition A,B has a perfect matching if and only if $|A| = |B|$ and every subset D of A satisfies*

$$|N(D)| \geq |D|$$

7.3.4 Theorem (p.192) *If G is a k-regular bipartite graph with $k \geq 1$, then G has a perfect matching.*

7.4 Edge Colouring

7.4.1 Theorem (p.194) *A bipartite graph with maximum degree Δ has an edge Δ-colouring.*

7.4.2 Lemma (p.194) *Let G be a bipartite graph having at least one edge. Then G has a matching saturating each vertex of maximum degree.*

An Application to Timetabling

7.4.3 Theorem (p.197) *Let G be a graph and k,m positive integers such that*

(a) G has an edge k-colouring;

(b) $q \leq km$

Then G has an edge k-colouring in which every colour is used at most m times.

7.4.4 Corollary (p.198) *In a bipartite graph G, there is an edge k-colouring in which each colour is used at most m times if and only if*

(a) $\Delta \leq k$, and

(b) $q \leq km$

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