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# Math Methods 12

## Portfolio Assignment 3

## SPEED LIMITS

1. The following piecewise equation may be used to compute Bill's distance *d*, measured in meters from his initial position at the traffic lights, at any time *t*, measured in seconds:

$$d(t) = \begin{cases} 0.85t^2, \; 0 \le t < 25 \\ 26.01t - 119, \; t \ge 25 \end{cases}$$

   a. Graphing the above function yields the following:

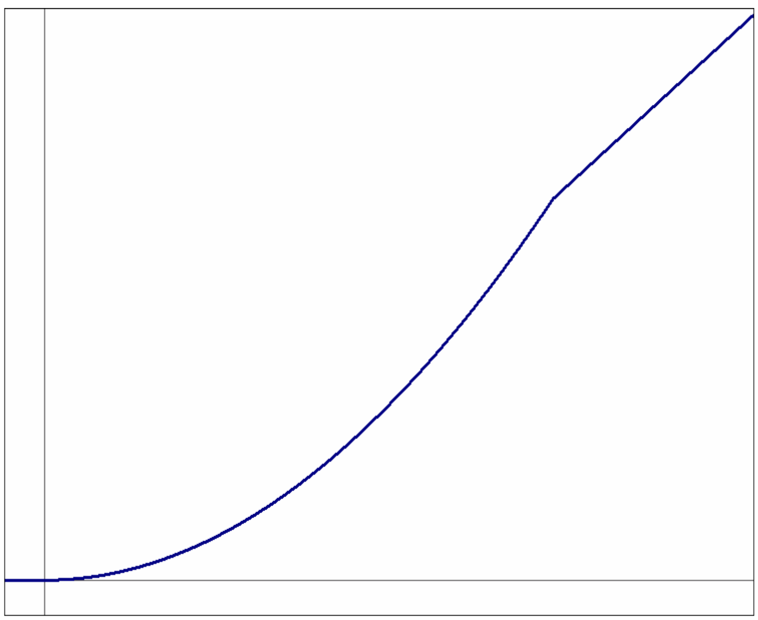


**Figure 1 Bill's distance as a function of time**

This distance function for Bill seems quite reasonable as it accounts for the fact that Bill would be required to accelerate for a period of time (25s) until he achieves a specific speed, after which he maintains a constant speed throughout the rest of the journey. This is what usually happens in reality, and so, generally speaking, the above distance vs. time function for Bill is well suited to model his distance over the 3 km stretch of highway.

Note: All graphs in this assignment have been generated with GraphCalc, freely available from http://www.graphcalc.com/.

b. The police radar is located at a distance of 340m from the traffic lights. Substituting this value into the given equation, and solving for *t* shows us that Bill passes the radar at a time $t = 20s$ :

$$t = \sqrt{\frac{\mathrm{d(t)}}{0.85}}$$
$$t = \sqrt{\frac{(340)}{0.85}}$$
$$t = 20s$$

In order to calculate the time *t* at which Bill gets pulled over, we use the second piece of the function, since the distance covered exceeds the range of the first piece:

$$t = \frac{d(t) + 119}{26.01}$$
$$t = \frac{(3000) + 119}{26.01}$$
$$t = 120s$$

c.

$$v = \frac{\Delta d}{\Delta t} \qquad v = \frac{\Delta d}{\Delta t}$$
$$v = \frac{\mathrm{d(20) - d(0)}}{20 - 0} \qquad v = \frac{\mathrm{d(120) - d(0)}}{120 - 0}$$
$$v = \frac{0.85(20)^2}{20} \qquad v = \frac{26.01(120) - 119}{120}$$
$$v = 17\ m/s \qquad v = 25.02\ m/s$$
$$v = 61.2\ km/h \qquad v = 90.07\ km/h$$

Bill's average velocity from 0 to 20 seconds is 61.2 km/h, while that from 0 to 120 seconds is 90 km/h. In neither case does he exceed the speed limit (Bill is allowed a maximum of 5 km/h above the specified limit).

d.

| Time(s) $t_o$ | $t_f$ | Δd | Δt | $v_{ave}$ (m/s) |
|---|---|---|---|---|
| 0 | 20 | 340.00 | 20 | 17.00 |
| 1 | 20 | 339.15 | 19 | 17.85 |
| 5 | 20 | 318.75 | 15 | 21.25 |
| 10 | 20 | 255.00 | 10 | 25.50 |
| 15 | 20 | 148.75 | 5 | 29.75 |
| 19 | 20 | 33.15 | 1 | 33.15 |
| 19.5 | 20 | 16.79 | 0.5 | 33.58 |
| 19.9 | 20 | 3.39 | 0.1 | 33.90 |

Note: Charts in this assignment have been generated with Microsoft Excel, and all calculations have been done using general spreadsheet formulas.

There is a purpose to completing the above chart: if we take a look at the computed average velocities, we see that they begin to approach a certain value as $\Delta t$ gets closer and closer to 0. This certain value is called the *limit* of the function at $t = 20s$. In the event that Δt becomes infinitely small (almost 0), the average velocity will converge to this limit. This limit is Bill's *instantaneous* velocity at $t = 20s$.

e.

| Time(s) $t_o$ | $t_f$ | h | d(t)-d(t-h) | $\frac{d(t)-d(t-h)}{t-(t-h)}$ |
|---|---|---|---|---|
| 19 | 20 | 1 | 33.15000 | 33.15 |
| 19.5 | 20 | 0.5 | 16.78750 | 33.575 |
| 19.9 | 20 | 0.1 | 3.39150 | 33.915 |
| 19.99 | 20 | 0.01 | 0.33992 | 33.9915 |
| 19.999 | 20 | 0.001 | 0.03400 | 33.99915 |
| 19.9999 | 20 | 1E-04 | 0.00340 | 33.999915 |
| 19.99995 | 20 | 5E-05 | 0.00170 | 33.9999575 |
| 19.99999 | 20 | 1E-05 | 0.00034 | 33.9999915 |
| 19.999999 | 20 | 1E-06 | 0.00003 | 33.99999914 |

From the above table, it is evident that as *h* gets smaller and smaller, the car's velocity – as computed using the expression $\frac{d(t)-d(t-h)}{t-(t-h)}$ – approaches 34 m/s. The smaller the value for *h* is, the closer the velocity is to 34 m/s. The numerical value 34 is therefore the limiting value for the velocity as *h* approaches 0. This can be described mathematically using the following notation:

$$\lim_{h \to 0} \frac{d(20)-d(20-h)}{t-(t-20)} = 40$$

f. Bill's velocity at exactly 20 seconds is called his **instantaneous velocity**. Ideally, in order to compute Bill's instantaneous velocity, the value of *h* would have to be to 0. However, if *h* equaled 0, we would end up with $\frac{0}{0}$, which is an indeterminate expression. To circumvent this problem, we take values for *h* as close to 0 as possible. The closer the value of *h* is to 0, the more accurate Bill's instantaneous velocity would be. For instance, a *h* of $1.0\times10^{-15}$ is so close to 0, that substituting this value into the above expression would give us a reasonable idea of what is happening at exactly 20 seconds. However, if we tried to simplify the above expression, we might be able to obtain a still better approximation at 20s:

$$v=\frac{d(t)-d(t-h)}{t-(t-h)}$$
$$v=\frac{d(20)-d(20-h)}{20-(20-h)}$$
$$v=\frac{0.85(20)^2-0.85(20-h)^2}{h}$$
$$v=\frac{0.85[20^2-(20-h)^2]}{h}$$
$$v=\frac{0.85[20^2-(20^2+h^2-40h)]}{h}$$
$$v=\frac{0.85(40h-h^2)}{h}$$
$$v=0.85(40-h),\ h\neq 0$$

With this simplified version of the quotient, we could substitute a very small number in place of *h* and get a clear picture of Bill's velocity at precisely 20s. In doing so, we obtain an instantaneous velocity of $34m/s$. It is to be noted that the equation $v=0.85(40-h)$ is valid only for $t=20s$, but is identical to the original quotient because no assumptions have been made during the simplification process.

g. It would be possible to find a generic function for Bill's velocity at any time *t*. We do so by following a similar technique used above. For the first 25s of Bill's journey:

$$v=\frac{d(t)-d(t-h)}{t-(t-h)}$$
$$v=\frac{0.85t^2-0.85(t-h)^2}{h}$$
$$v=\frac{0.85t^2-0.85t^2-0.85h^2+1.7th}{h}$$
$$v=1.7t-0.85h$$
$$v=0.85(2t-h)$$

It is to be noted that the slope of a distance vs. time function is nothing but velocity. Therefore, we could use this concept of "slope" to easily construct a function for Bill's velocity. So if we were to differentiate $d(t)$ with respect to *t*, we could find a function $d'(t)$ that could give us the slope for any time *t:*

$$
\begin{aligned}
v_{\text{inst}} &= \frac{d}{dt}[0.85t^2] \\
v_{\text{inst}} &= 0.85\frac{d}{dt}[t^2] \\
v_{\text{inst}} &= 0.85(2\times t^{(2-1)}) \\
v_{\text{inst}} &= 1.7t,\ \ 0 \le t < 25
\end{aligned}
$$

and for 25s and above:

$$
\begin{aligned}
v_{\text{inst}} &= \frac{d}{dt}[26.01t - 119] \\
v_{\text{inst}} &= \frac{d}{dt}[26.01t] - \frac{d}{dt}[119] \\
v_{\text{inst}} &= 26.01\frac{d}{dt}[t] - 0 \\
v_{\text{inst}} &= 26.01,\ \ t \ge 25
\end{aligned}
$$

The above two functions may be combined by defining them piecewise. Thus,

$$
v_{\text{inst}} = d'(t) = \begin{cases} 1.7t, & 0 \le t < 25 \\ 26.01, & t \ge 25 \end{cases}
$$

If we tested the above function with $t = 20$, we get a corresponding value of 34 m/s, which agrees with our previous observations.

2. It is difficult to say whether Bill should or shouldn't have been issued a speeding ticket. In part b) of question 1, we found that Bill passes the radar at $t = 20s$, and the instantaneous velocity at that moment was found to be 34 m/s, which corresponds to a speed of 122.4 km/h. This speed is well above the speed limit, and according to the radar which measures instantaneous velocity (as opposed to average velocity), Bill is over-speeding. The radar itself has made **no error** whatsoever in its calculations; however, Bill himself sees only an average velocity on his car's speedometer, and Bill has always remained within the **average** speed limit of 90 km/h. Given these circumstances and Bill not able to know his instantaneous velocity, it could be argued that Bill **should not** have been issued with a speeding ticket.

   Mathematically however, one could possibly argue that Bill's instantaneous velocity exceeds the limit of 90 km/h, and that it was appropriate for Bill to have been issued with a ticket. It is therefore difficult to make a just and judicial decision without additional information about the situation.

3. If we create a continuous for Art, his initial distance would 10, because Bill started approximately 10m behind Art at the traffic lights. Therefore, $A(0) = 10$.

   If Bill were to never pass Art for the entire 3 km stretch of highway, it would mean that the slope of Art's continuous function must be the same or equal to that of Bill's (during the latter part of the journey, when his velocity becomes constant). In other words, A(t) must always be greater than d(t) for any *t*. Graphically, this would mean that Bill's distance function would always be below that of Art's for any *t*, and would never intersect Art's.

   If Art were to maintain a constant velocity throughout the entire 3 km of highway, his maximum velocity would be the slope of his linear distance-time function. However, if he were to travel at 26.01m/s throughout, he would definitely violate the speed limit as 26.01m/s corresponds to 93.6 km/h (ignoring the small allowance of 5 km/h). If Art traveled at 25m/s all through the highway, he would never violate the speed limit, although his slope would now be smaller than that of Bill's during the latter half of the journey. This wouldn't pose a problem as Bill would still not be able to catch up with Art before the 3 km highway came to an end.

   Keeping the above discussed points in mind, the simplest function that would also be continuous everywhere would be a linear one described mathematically as follows:

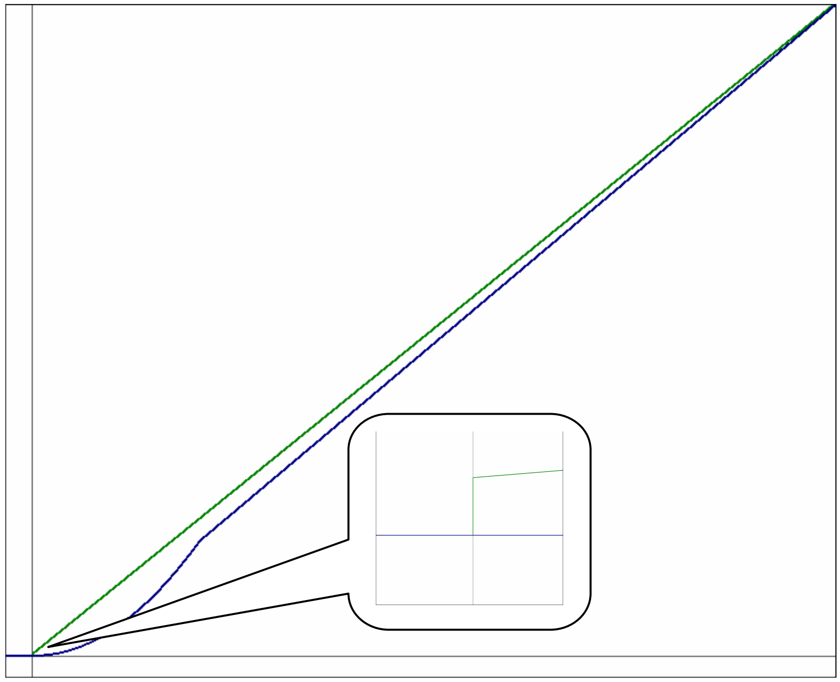
$$\mathrm{A(t)} = 25x + 10$$

   Using the above equation, Art would still be 8m ahead of Bill at the end of the highway (and hence does not violate condition 3):

$$A(d^{-1}(3000)) = 3008$$

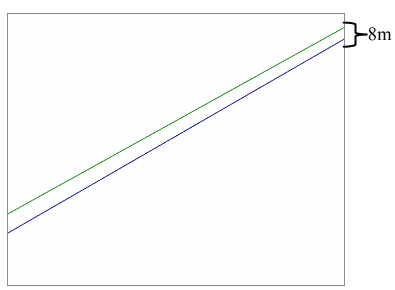
   This would then mean that Bill eventually overtakes Art at some point in time (after the 3 km highway), but accounting for that is outside the scope of condition 3.

4.



The graph adheres to the conditions imposed on Art's distance function:

- The condition A(0) = 10 has been satisfied by setting the y-intercept for Art's function equal to 10. Art's function (green) now starts at 10m when $t = 0$.
- Art would never violate the speed limit no matter where the police radar is placed because his velocity is always constant at 25 m/s, which corresponds to 90 km/h. His instantaneous velocity would also therefore be 90 km/h at any time *t*.
- Even at the end of the highway, Art is still 8m ahead of Bill, as can be seen from the enlarged portion below.




5. In order to comment on the physical ramifications of Bill's $d(t)$ and Art's $A(t)$, we would be required to have a basic understanding of real-world situations, acceleration, and equations of kinematics.

   Bill's distance function d(t) **does not** seem to be reasonable due to the following reason: The first piece of the function $0.85t^2$ tells us that Bill accelerates from rest for 25 seconds. With the aid of equations of motion, it can be shown that Bill's average acceleration is $0.85\times 2=1.7m/s^2$. If he accelerated at this rate for 25s, he would reach a speed of $1.7\times 25=42.5m/s=153km/h$ at the end of the 25s period. Since Bill is a careful driver, it does not seem plausible that Bill would accelerate to such a high speed (which is well above the allowed limit) before lifting his foot off the pedal. A much better distance function for Bill (assuming Bill stops accelerating once his speedometer reads 90 *km/h*) would be:

$$d(t)=\begin{cases}0.5t^2, \ 0\le t<25\\ 25t-312.5, \ t\ge 25\end{cases}$$

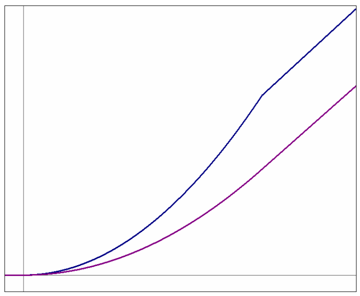


**Figure 2 Bill's ideal distance graph**

The corresponding graph (purple) can be seen in Figure 2. The original graph (blue) has been plotted for comparison.

On the same note, Art's distance function A(t) **does not** seem to be physically acceptable either. In real-world situations, cars always accelerate to a certain speed before coasting at a constant speed. This initial acceleration is required in order to pick up some speed. According to the linear function described for Art, it seems that Art picks up 25m/s (90 km/h) is just a fraction of a second, which in reality does not seem possible.

Thus we conclude that although Bill's and Art's functions are a good model of their distances over an interval of time, they are not physically realistic and cannot be used to compute their actual positions or velocities and at a given time *t*.