Math Methods 12

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Portfolio Assignment 4 – Type II

Analysis of a Quartic Function

1. The general formula for the binomial expansion of $(x+y)^n$ is given by

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + nxy^{n-1} + y^n$$

Substituting $x = a$, $y = b$ and $n = 3$ yields

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \qquad (1)$$

The same result may be obtained by multiplying out the factors:

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

2. Letting $a = 2$ and $b = -x$ in (1) yields

$$\begin{aligned}(2-x)^3 &= (2)^3 + 3(2)^2(-x) + 3(2)(-x)^2 + (-x)^3\\ &= 8 - 12x + 6x^2 - x^3\end{aligned}$$

3. Using the result from question 2,

$$\begin{aligned}x(2-x)^3 &= x(8-12x+6x^2-x^3)\\ &= 8x - 12x^2 + 6x^3 - x^4\end{aligned}$$

Re-arranging the terms in decreasing degrees, we obtain $x(2-x)^3 = -x^4 + 6x^3 - 12x^2 + 8x$ which is of the form $px^4 + 6x^3 - 12x^2 + qx$. Comparing coefficients we find $p = -1$ and $q = 8$.

4.
$$\begin{aligned}y &= x(2-x)^3\\ y &= -x^4 + 6x^3 - 12x^2 + 8x\\ \therefore \frac{dy}{dx} &= -4x^3 + 18x^2 - 24x + 8\end{aligned}$$

5. According to ***the chain rule***, $(f \circ g)'(x) = f'(g(x))g'(x)$. Alternately, if $y = f(g(x))$ and $u = g(x)$ then $y = f(u)$ and $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Since the function $y = (2 - x)^3$ is a composition of two functions, namely $f(x) = (2 - x)$ and $g(x) = x^3$, we *must* obey the chain rule while performing the differentiation.

If we let u represent the factor $2 - x$ in $y = (2 - x)^3$, we have

$$
\begin{aligned}
y &= (2-x)^3 \\
y &= u^3 \\
\frac{dy}{dx} &= 3u^2 \cdot \frac{d}{dx}[2-x] \\
&= 3(2-x)^2 \cdot (-1) \\
&= -3(2-x)^2
\end{aligned}
$$

The student's result $3(2 - x)^2$ is incorrect because he has failed to multiply his result by the derivative of the "inside" function, namely $(2 - x)$. In doing so, he has inadvertently lost the multiplicand -1 which is the derivative of $(2 - x)$.

6. We obtain the same result as question 4 if we use a combination of product and chain rules to differentiate $y = x(2 - x)^3$:

$$
\begin{aligned}
y &= x(2-x)^3 \\
\frac{dy}{dx} &= (2-x)^3 \frac{d}{dx}[x] + x \frac{d}{dx}[(2-x)^3] \\
&= (2-x)^3 + x[3(2-x)^2 \cdot (-1)] \\
&= -4x^3 + 18x^2 - 24x + 8
\end{aligned}
$$

7. $y = x(2 - x)^3$

From questions 4 and 6, we have $\frac{dy}{dx} = -4x^3 + 18x^2 - 24x + 8$.

The second derivative $\frac{d^2y}{dx^2}$ can be computed by differentiating $\frac{dy}{dx}$.

So, $\frac{d^2y}{dx^2} = -12x^2 + 36x - 24$

When $x = 2$, $y = (2)[2 - (2)]^3 = 0$.

Similar substitutions yield $\left.\frac{dy}{dx}\right|_{x=2} = -4(2)^3 + 18(2)^2 - 24(2) + 8 = 0$

and $\left.\frac{d^2y}{dx^2}\right|_{x=2} = -12(2)^2 + 36(2) - 24 = 0$.

Conclusions:

- Since $y = 0$ when $x = 2$, 2 is a zero of the function $f(x) = x(2-x)^3$. Consequently, the graph of *f* must touch the *x*-axis at this point.
- Since $\frac{dy}{dx} = 0$ when $x = 2$, this point is a ***stationary point*** of *f*. In other words, the tangent line at $x = 2$ has gradient 0 and is thus horizontal.
- Since $\frac{d^2y}{dx^2} = 0$ when $x = 2$, it follows that *f* has an ***inflection point*** at $x = 2$. In other words, *f* changes the direction of its concavity at the point (2, 0).
- Since $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$, it is impossible to say, at this point in time, whether *f* has a relative maximum, a relative minimum, or neither at $x = 2$ by the second derivative test.

8. A rough sketch of $y = x(2-x)^3$ can be seen in Figure 1. The conclusions reached in the previous question are consistent with the graph of *f*.

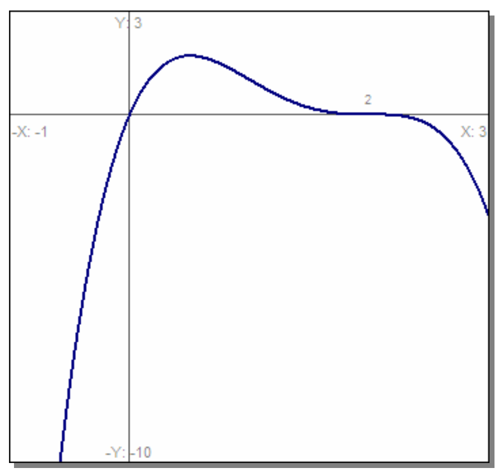


Figure 1 $y = x(2-x)^3$

The last conclusion as to whether *f* has a relative maximum, a relative minimum, or neither deserves further consideration. Although the second derivative test is inconclusive since $\frac{dy}{dx} = \frac{d^2y}{dx^2} = 0$, we note that $x = 2$ is a root of *f* and has ***multiplicity*** 3. This is because $(x-2)^3$ divides the polynomial f but $(x-2)^4$ does not. There is a close relationship between the multiplicity of a root of a polynomial and the behavior of the graph in the vicinity of the root. Since the multiplicity $m = 3$ of the root $x = 2$ is *odd* (indivisible by 2), the graph will be tangent to the *x*-axis at $x = 2$, will cross the *x*-axis, and will also have an inflection point there. We have already recognized $x = 2$ as being both a stationary as well as an inflection point. However, the fact that the graph crosses the *x*-axis at this point indicates that *f* has neither a relative maximum nor a relative minimum there. In order for *f* to have a relative extremum at $x = 2$, the graph will have to "bounce-off" the *x*-axis once it touches it. This conclusion is evident in the graph of *f* in Figure 1.

9. A sketch graph of the definite integral $\int_0^2 x(2-x)^3 dx$ can be seen in Figure 2. The area enclosed by the graph and the *x*-axis has been shaded and computed to equal 1.6.

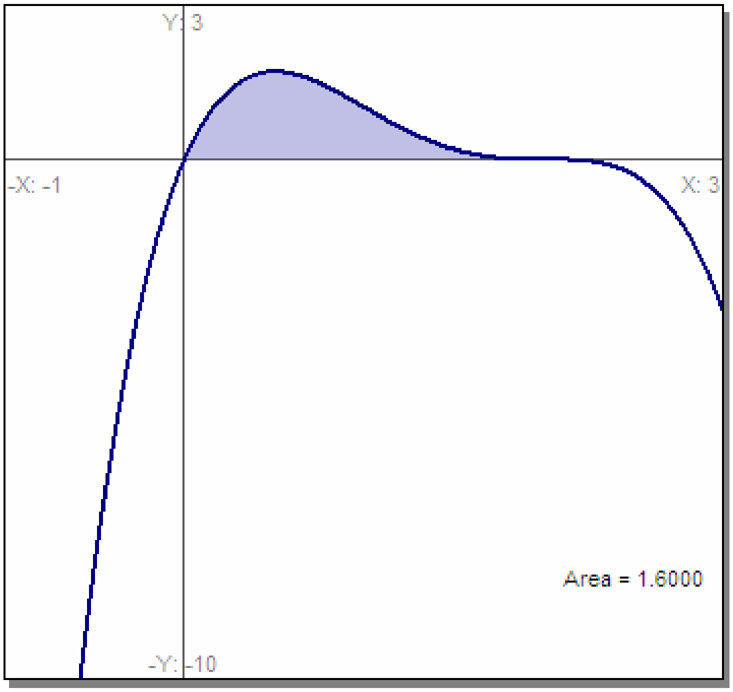


Figure 2 $y = \int_0^2 x(2-x)^3 dx$

Note: All graphs in this assignment, unless otherwise noted, were generated with GraphCalc, freely available from http://www.graphcalc.com/.

10. By the method of *u*-substitution, it is possible to evaluate the definite integral $\int_0^2 x(2-x)^3 dx$ without needing to expand the cubed term.

If we make the substitution $u = 2 - x$, it follows that $du = -dx$ or $dx = -du$. This leaves a factor of x unresolved in the integrand. However, since $u = 2 - x$, we have $x = 2 - u$.

With this substitution

if $x = 0$, $u = 2 - 0 = 2$
if $x = 2$, $u = 2 - 2 = 0$

so

$$\begin{aligned}\int_0^2 x(2-x)^3\,dx &= \int_2^0 (2-u)u^3(-du)\\ &= -\int_2^0 (2u^3-u^4)\,du\\ &= \int_0^2 (2u^3-u^4)\,du\\ &= \left[\frac{u^4}{2}-\frac{u^5}{5}\right]_{u=0}^2\\ &= 2^3-\frac{2^5}{5}\\ &= \frac{8}{5}\\ &= 1.6\end{aligned}$$

which matches the result obtained in question 9.

> **REMARK.** The evaluation of the above definite integral is not only tedious but also error-prone since it was performed manually by hand. The same computation was done by the Maxima[1] Computer Algebra System (CAS) in less than a millisecond:
>
> ```
> rajesh@rbox ~ $ maxima
> Maxima restarted.
> (%i1) integrate(x*(2-x)^3, x, 0, 2);
>
>                     8
> (%o1)               -
>                     5
> ```

11. In question 3, we showed that $x(2-x)^3 = -x^4 + 6x^3 - 12x^2 + 8x$. Integration of this quartic polynomial is as follows:

[1] http://maxima.sourceforge.net/

$$\int_0^2 x(2-x)^3\,dx = \int_0^2 (-x^4+6x^3-12x^2+8x)\,dx$$
$$= \left[-\frac{x^5}{5}+6\frac{x^4}{4}-12\frac{x^3}{3}+8\frac{x^2}{2}\right]_0^2$$
$$= \left[-\frac{x^5}{5}+\frac{3x^4}{2}-4x^3+4x^2\right]_0^2$$
$$= -\frac{(2)^5}{5}+\frac{3(2)^4}{2}-4(2)^3+4(2)^2$$
$$= -\frac{32}{5}+24-32+16$$
$$= \frac{40-32}{5}$$
$$= 1.6$$

which once again agrees with the result obtained in questions 9 and 10.

12. The ***trapezoidal approximation*** of a definite integral is an average of the left-hand and right-hand approximations. This approximation is given by

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{2n}\right)\left[y_0+2y_1+\cdots+2y_{n-1}+y_n\right]$$

With $f(x)=x(2-x)^3$, $a=0$, $b=2$ and $n=4$ the definite integral may be approximated as follows (the x values for the trapezoids being incremented by $\frac{2-0}{4}=0.5$ each time):

$$\int_0^2 x(2-x)^3\,dx \approx \left(\frac{2-0}{2\times 4}\right)\left[f(0)+2\big(f(0.5)+f(1)+f(1.5)\big)+f(2)\right]$$
$$\approx \frac{1}{4}\left[0+2(1.6875+1+0.1875)+0\right]$$
$$\approx 1.4375$$

The ***absolute error*** (or margin of error) in this approximation is given by

$$|E_T| = \left|\int_a^b f(x)\,dx - T_n\right|$$

Using this, we find the error in approximation to be $|1.6-1.4375|=0.1625$ or

$$\frac{0.1625}{1.6}\times 100\,\% \approx 10.2\%$$

13. The easiest way to find out the number of solutions to the equation $x(2-x)^3 = 1$ would be to graph the function $y = x(2-x)^3 - 1$ and count the number of intersections with the *x*-axis. Since this is a quartic function, we expect to have *at most* 4 solutions. The graph of $y = x(2-x)^3 - 1$ is shown in Figure 3.

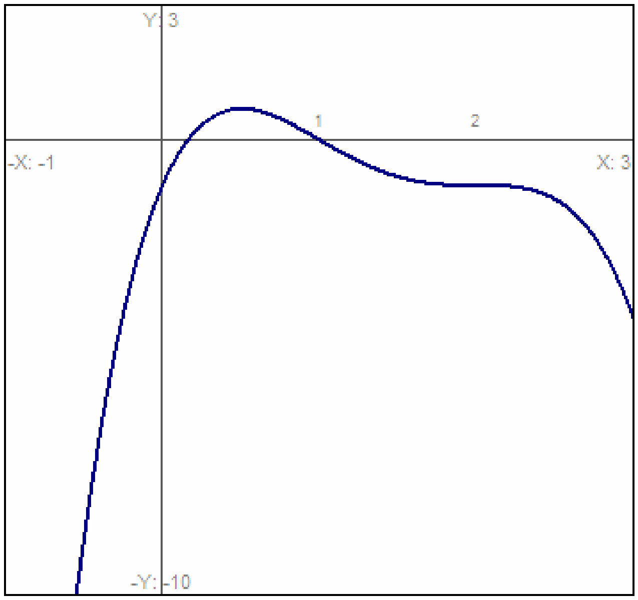
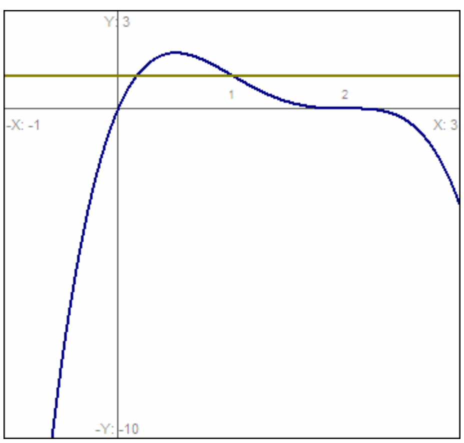


Figure 3 $y = x(2-x)^3 - 1$

It is evident from the above graph that the equation $x(2-x)^3 = 1$ has exactly two real solutions.



Another method to determine the number of solutions would be to graph the function $y = 1$ over the graph of $y = x(2-x)^3$ that was graphed in Figure 1 and count the number of intersections (see Figure 4). It can be seen that the two curves intersect twice and so the equation $x(2-x)^3 = 1$ has exactly two real solutions.

Figure 4 $y_1 = 1$
$y_2 = x(2-x)^3 - 1$

REMARK. If a graphing utility is unavailable, the Maxima CAS can give us the number of solutions quickly, thanks to the built-in NROOTS() or REALROOTS() functions. For instance:

```
(%i1) REALROOTS(x*(2-x)^3-1);

                                          5392641
(%o1)                               [x = --------, x = 1]
                                          33554432
```

We find two real roots for this equation. To confirm, we could use the NROOTS() function.

```
(%i2) NROOTS(x*(2-x)^3-1);

(%o2)                                      2
```

Since we want all roots between $x=-1$ and $x=3$, we specify the lower and upper bounds. However, since both solutions are between these bounds, we still obtain 2.

```
(%i3) NROOTS(x*(2-x)^3-1, -1, 3);

(%o3)                                      2
```

The equation $x(2-x)^3=1$ can be re-arranged in two ways:

- Dividing both sides by $(2-x)^3$ [we can do this since $x \neq 2$], we get $x=\dfrac{1}{(2-x)^3}$.

  If x equaled $\dfrac{1}{(2-x)^p}$, then $p=3$.

- $$x(2-x)^3=1$$
  $$(2-x)^3=\frac{1}{x}$$
  $$(2-x)=\sqrt[3]{\frac{1}{x}}$$
  $$x=2-x^{-\frac{1}{3}}$$

where the second step is justified by recognizing the fact that $x \neq 0$. If x equaled $q-(x)^{-\frac{1}{3}}$, then $q=2$.

Using the methods of simple iteration and taking $x_0=0.5$:

$$x_1 = \frac{1}{\left[2-(0.5000)\right]^3} = 0.2963$$

$$x_2 = \frac{1}{\left[2-(0.2963)\right]^3} = 0.2022$$

$$x_3 = \frac{1}{\left[2-(0.2022)\right]^3} = 0.1721$$

$$\vdots$$

$$x_8 = \frac{1}{\left[2-(0.1608)\right]^3} = 0.1607$$

$$x_1 = 2-(0.5000)^{-\frac{1}{3}} = 0.7401$$

$$x_2 = 2-(0.7401)^{-\frac{1}{3}} = 0.8945$$

$$x_3 = 2-(0.8945)^{-\frac{1}{3}} = 0.9621$$

$$\vdots$$

$$x_{10} = 2-(0.9999)^{-\frac{1}{3}} = 1.0000$$

These two results (accurate to within 4 decimal places) are approximations of the solutions to the equation $x(2-x)^3 = 1$ as can be seen in Figure 3. Each form of the original equation $x(2-x)^3 = 1$, namely $x = 1/(2-x)^3$ and $x = 2 - x^{-1/3}$, yields one of the two solutions.

Using other values for x_0, we find that the results converge to the same answer every time. This convergence can be seen graphically in Figure 5.

It is helpful to remind ourselves of what actually happens as we perform the iteration. Each term of the iterative sequence x_n is computed from the previous term using the iterative function. Mathematically, this means that $x_{n+1} = f(x_n)$, where $f(x)$ is any one of the above two iterative functions. If this process is repeated continuously and the limit taken, we will eventually obtain a solution to the original equation. This is to say that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n)$, where $f(x)$ is an iterative function (the re-arranged equation solved for *x*). In our case,

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{(2-x_n)^3} \approx 0.1607 \text{, and}$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} 2-(x_n)^{-\frac{1}{3}} = 1.$$

The ***Newton-Raphson*** method of approximation is given by the following iterative function:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \qquad (2)$$

In order to determine if the Newton-Raphson method can be used to yield the same results, we check to see whether the absolute value of the derivative evaluated at some test value x_0 is in between 0 and 1. In other words, if $f(x)$ is a differentiable function

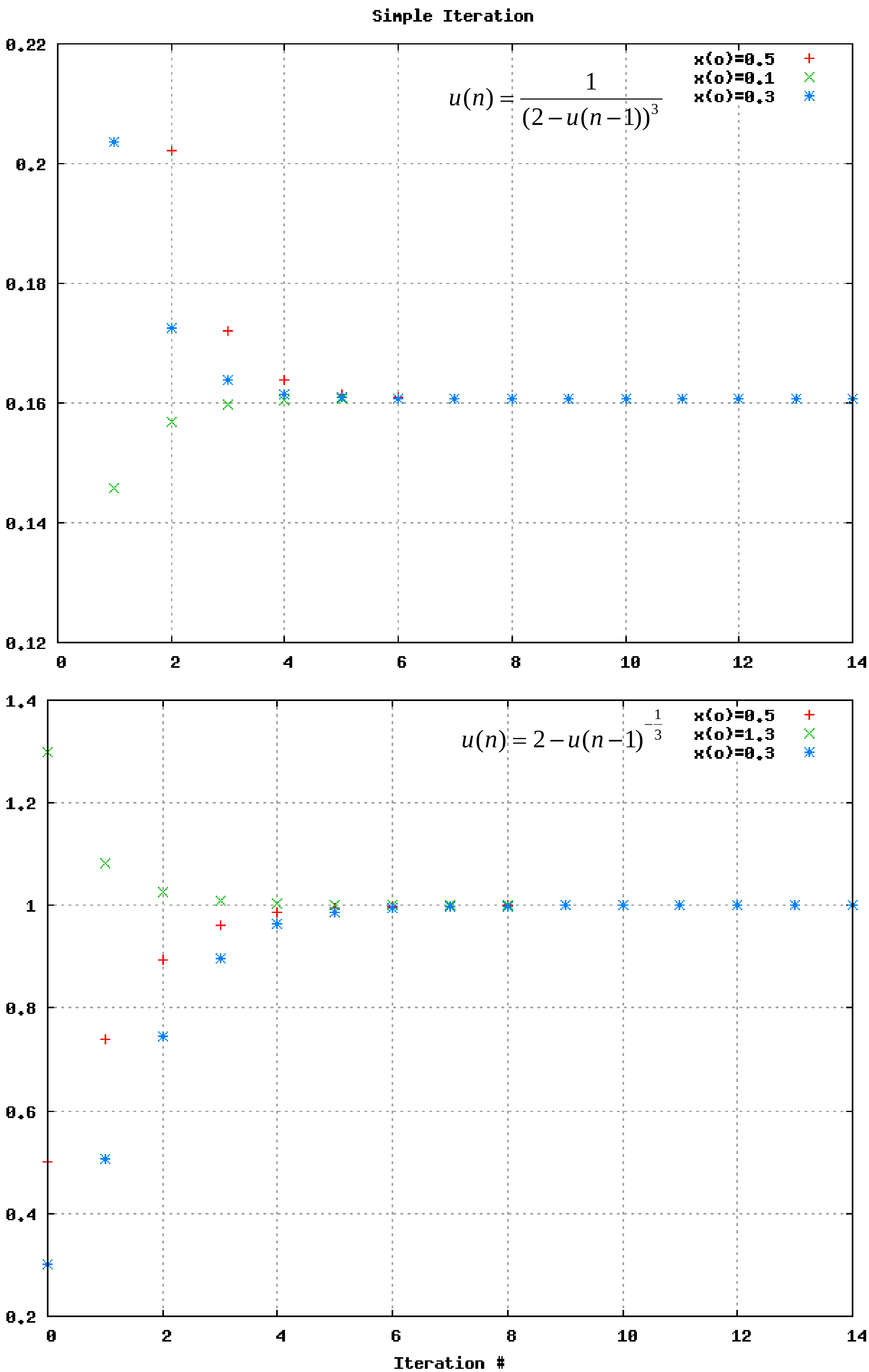

Simple Iteration
$u(n) = \frac{1}{(2-u(n-1))^3}$
x(o)=0.5
x(o)=0.1
x(o)=0.3
0.22
0.2
0.18
0.16
0.14
0.12
0
2
4
6
8
10
12
14
$u(n) = 2-u(n-1)^{-\frac{1}{3}}$
x(o)=0.5
x(o)=1.3
x(o)=0.3
1.4
1.2
1
0.8
0.6
0.4
0.2
Iteration #

Figure 5

Note: The graphs in Figure 5 were generated with Gnuplot, freely available from http://www.gnuplot.info/.

and we started with some test value x_0, then $|f'(x_0)|$ must be less than 1 for the iterative function to converge and yield an approximation. If this condition is not satisfied, we will have a diverging iterative sequence and will never reach an approximation.

Furthermore, if $f'(x_0)$ is positive, we will have a ***monotonic convergence***. Conversely, if $f'(x_0)$ is negative, we will have an ***oscillating convergence***. The meaning of these two terms is left unexplained for the interested to pursue.

Since we are attempting to solve the equation $x(2-x)^3 = 1$, we let $f(x) = x(2-x)^3 - 1$, so $f'(x) = -4x^3 + 18x^2 - 24x + 8$ (verify) and (2) becomes

$$x_{n+1} = x_n - \frac{-x^4 + 6x^3 - 12x^2 + 8x - 1}{-4x^3 + 18x^2 - 24x + 8}, \text{ or } x_{n+1} = x_n + \frac{x^4 - 6x^3 + 12x^2 - 8x + 1}{4x^3 - 18x^2 + 24x - 8} \quad (3)$$

If we started with a test value $x_0 = 0.5$, then

$$f'(0.5) = -4(0.5)^3 + 18(0.5)^2 - 24(0.5) + 8 = 0$$

and can therefore *not* be used to give a result since the denominator in (3) evaluates to 0 and is thus undefined.

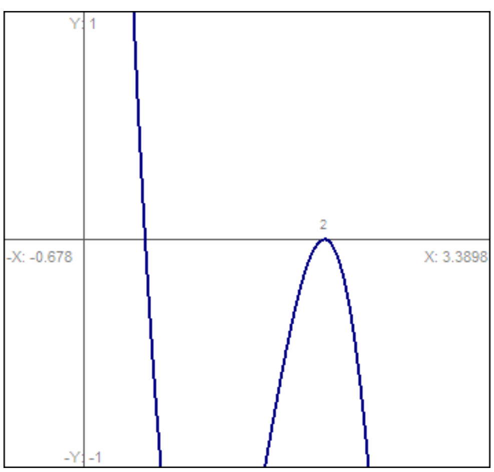


Figure 6 $y = -4x^3 + 18x^2 - 24x + 8$

If we used different test values such as $x_0 = 0.1, 0.3, 1.3$, we still find that we are unable to use the Newton-Raphson method since in each case $|f'(x_0)| > 1$. For what value of x_0 will $|f'(x_0)|$ be less than 1 but not equal to 0? The answer to this problem can be obtained by graphing $f'(x)$ [see Figure 6] and looking for values of *x* whose *y* values are not zero and are in the interval (-1, 1). For this, we solve the equations $f'(x) = 1$ and $f'(x) = -1$. This is a straight-forward task that can be achieved with the Maxima CAS:

```
(%i1) f(x):= x*(2-x)^3-1;
(%i2) define(df(x), expand(diff(f(x), x)));
```

```
(%i3) realroots(df(x)-1);

                                          13491845
(%o3)                                [x = --------]
                                          33554432

(%i4) realroots(df(x)+1);

                               21272657      3      79390639
(%o4)                     [x = --------, x = -, x = --------]
                               33554432      2      33554432
```

$\frac{13491845}{33554432} \approx 0.403, \frac{21272657}{33554432} \approx 0.634, \frac{79390639}{33554432} \approx 2.366$

We can thus conclude that in order to be able to solve the equation $x(2-x)^3 = 1$ using the Newton-Raphson method, we *must* use an initial value that is in the interval $(0.403, 0.500) \cup (0.500, 0.634) \cup (1.500, 2.000) \cup (2.000, 2.366)$.

Using any initial value from this interval and (2), we obtain the same solutions to the equation $x(2-x)^3 = 1$, namely $x = 0.1607$ and $x = 1$ (verify).