Interesting Integrals

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1 The Factorial

We begin by defining the factorial of an arbitrary number $n \in \mathbb{R}$.

For a regular *natural* number $n \in \{0, 1, 2 \ldots\}$, we define the factorial as being

$$n! = n(n-1)! = n(n-1)(n-2)\cdots 1$$

where we've been thought by rote, without explanation, that $0! = 1$. As far as we know, factorials did not exist for non-real, negative, fractional or decimal quantities.

However, Euler noticed that

$$\begin{aligned}
\int_0^\infty xe^{-x}dx &= \lim_{b\to\infty}\int_0^b x\frac{d}{dx}(-e^{-x})dx \\
&= \lim_{b\to\infty}\left[-xe^{-x}|_0^b - \int_0^b -e^{-x}dx\right] \\
&= \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^b \\
&= \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + 1\right] \\
&= 0 - 0 + 1 \\
&= 1 = 1!
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty x^2e^{-x}dx &= \int_0^\infty x^2\frac{d}{dx}(-e^{-x})dx \\
&= \lim_{b\to\infty}\left[x^2 \cdot -e^{-x}\right]_0^b - \int_0^\infty -e^{-x} \cdot 2xdx \\
&= \lim_{b\to\infty}\left[-b^2e^{-b}\right] + 2 \\
&= 2 \\
&= 2 \cdot 1 = 2!
\end{aligned}$$

$$\begin{aligned}
\int_0^\infty x^3e^{-x}dx &= \lim_{b\to\infty}\left[x^3 \cdot -e^{-x}\right]_0^b - \int_0^\infty -e^{-x} \cdot 3x^2dx \\
&= \lim_{b\to\infty}\left[-b^3e^{-b}\right] + 3 \cdot 2 \\
&= 6
\end{aligned}$$

$$= \quad 3 \cdot 2 \cdot 1 = 3!$$

Euler tried to interpolate and finally hit upon the creation of the following function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}$$

For x integer

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt = (n-1)!$$

and

$$\Gamma(n+1) = \int_0^\infty t^{n+1-1} e^{-t} dt = n!$$

which formally defines the factorial.

Thus $\underbrace{\Gamma(n+1)}_{n!} = \underbrace{n\Gamma(n)}_{n(n-1)!}$, and is called the recurrence relation for the gamma function.

So

$$\Gamma\left(\frac{1}{2}\right) \quad = \quad \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \cdots = \sqrt{\pi}$$

$$\begin{aligned}
\Gamma\left(\frac{3}{2}\right) &= \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi} \\
\Gamma\left(\frac{5}{2}\right) &= \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\frac{\sqrt{\pi}}{2} = \frac{3\sqrt{\pi}}{4}
\end{aligned}$$

2 The Interesting Integral

We can now proceed to integrate

$$\int_0^\infty x e^{-x^3} dx$$

Let $u = x^3 = x^2 x$, so $du = 3x^2 dx$ or $dx = \frac{du}{3x^2} = \frac{du}{3\frac{u}{x}} = \frac{xdu}{3u}$

Hence

$$\begin{aligned}
\int_0^\infty x e^{-x^3} dx &= \int_0^\infty u^{1/3} e^{-u} dx \\
&= \frac{1}{3}\int_0^\infty u^{-2/3} e^{-u} x du \\
&= \frac{1}{3}\int_0^\infty u^{-2/3} e^{-u} u^{1/3} du \\
&= \frac{1}{3}\int_0^\infty u^{-1/3} e^{-u} du
\end{aligned}$$

$$= \frac{1}{3}\int_0^\infty u^{\frac{2}{3}-1}e^{-u}du$$
$$= \underline{\underline{\frac{1}{3}\,\Gamma\left(\frac{2}{3}\right)}}$$

3 Discussion

```
Date: Fri, 02 Dec 2005 17:42:38 -0500
From: Rajesh Kumar <rajesh@meetrajesh.com>
User-Agent: Mozilla Thunderbird 1.0.6 (Windows/20050716)
To: Ted <tmamo@engmail.uwaterloo.ca>, zhu chen <uhznehc1987@hotmail.com>
Subject: Lovely Integrals

Hey Ted/Clarke,

I tried the x*e^(-x^3) integral on Maxima, a powerful computer
algebra system and a free alternative to Maple. Turns out our
solution was right.

(%i2) integrate(x*%e^(-x^3),x);

                           /        3
                           [     - x
(%o2)                      I x %e      dx
                           ]
                           /


As you can see, Maxima can't do the indefinite integral. Neither
can I.

(%i3) integrate(x*%e^(-x^3),x,0,inf);

                                 2
                           gamma(-)
```

```
                                      3
(%o3)                          --------
                                    3
```

But it CAN do the improper integral from 0 to infinity and the answer is the same as that obtained by us in class today.

But GAMMA(2/3) seems unexpandable. I tried the following:

```
(%i26) gamma(1/2);

(%o26)                                  sqrt(%pi)
(%i8) gamma(-1/2);

(%o8)                                 - 2 sqrt(%pi)
```

and they both work. But

```
(%i27) gamma(2/3);

                                            2
(%o27)                                 gamma(-)
                                            3
```

spits back the same answer. We might have to wait till MATH 119 to actually solve this.

However, MAPLE does the original integral amazingly fast, but gives me the answer in terms of the WhittakerM() function, which is not very useful to us.

However, MATLAB, spits out the right numerical answer:

```
>> gamma(2/3)
ans =
     1.3541
```

Conclusion: MATLAB rules!

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RKS

Date: Sat, 03 Dec 2005 07:39:42 -0500
From: tmamo@engmail.uwaterloo.ca
To: Rajesh Kumar <rajesh@meetrajesh.com>
Subject: Re: Lovely Integrals
User-Agent: Internet Messaging Program (IMP) 3.1 / FreeBSD-4.6.2

Hello Rajesh:

That's really a wonderful result. Thanks to Tenti who showed us the Gamma funciton which we were thinking to be something difficult. I think I will try to solve Gamma(2/3) during the holidays. We should be able to solve it during the two weeks.

I also tried it in Matlab and I get the same answer. I was also wondering what 1.3541 will be in terms of fractions or pi or something. I will think about that too. Now I have to finish the Algebra project.

Thanks for the info. I enjoy our educational discussion very much!

Bye!
